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Optimization of urban paths in pandemic era

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Abstract

This work focuses on a possible redistribution of car traffic, modelled via a fluid dynamic approach, within a part of the Caltanissetta city (Italy), when critical events, such as gathering phenomena, occur. Via a decentralized approach, a cost functional, that indicates the asymptotic average velocity of police cars, is maximized with respect to traffic parameters at nodes with two incoming and two outgoing roads. Then, the management of high traffic is studied by local optimal coefficients at each node of the network. The whole traffic phenomena are analyzed by simulations, which confirm the correctness of the optimization procedure. It is also proved that optimal parameters manage a fast transit of police cars on assigned paths on the network.

Keywords: Conservation laws; Gathering phenomena; Optimization.

1. Introduction

Road networks are often interested by phenomena of various types, namely possible queue formations, pollution, long travel times, and so on. In most cases, high traffic levels lead to unsuitable situations, for instance car accidents. Nowadays, considering the national emergency conditions due to COVID-19, gathering phenomena are also possible. This last aspect is highly critical as it could inevitably lead to an increase in the levels of contagion. In this sense, law agents must try to avoid assemblies by fast interventions in critical urban places. This suggests to adopt methodologies for road traffic in emergency and/or critical conditions. This work considers some optimization results for a part of Caltanissetta urban network, Italy, for the redistribution of car flows so that police cars could cross assigned roads at the maximum possible speed in order to avoid gathering phenomena.

A fluid dynamic model is considered where the dynamics of car densities on roads is modelled by conservation laws ((Coclite et al., 2005), (Lighthill and Whitham, 1955), (Richards, 1956)), while dynamics at $n \times m$ junctions (namely nodes with n incoming roads and m outgoing roads) are studied by rules of traffic distributions and right of ways (if $n > m$). As for the distribution coefficients as control parameters, we aim to redirect traffic at 2×2 nodes in order to manage emergency and/or critical situations. Hence, assuming that police vehicles could cross assigned paths, a cost functional $V_{(u,v)}$, that represents, for 2×2 nodes, the average velocities of car polices on the incoming road I_u, $u \in \{1, 2\}$, and on the outgoing road I_v, $v \in \{3, 4\}$, is considered. The optimization procedure gives the distribution parameters that maximize the functional and allow a fast transit of police cars to reach the places of gathering phenomena.

As, for complex networks, the analysis of $V_{(u,v)}$ is a very hard task, a decentralized methodology is adopted, namely: the asymptotic dynamics (for very large times) is adopted and an exact solution for $V_{(u,v)}$ is obtained for a node of 2×2 type. Then, we construct a global (sub)optimal solution for the whole network by applying simply the obtained local optimal solutions at each node with two incoming roads and two outgoing roads. Similar studies have been also considered for other road nodes and various functionals, see (Cascone et al., 2007), (Cascone et al., 2008), and (D'Apice et al., 2011), as well as different numerical approaches are described in (Tomasiello, 2011) and (Tomasiello, 2012). Similar topics, that deal with fluid dynamic models, are considered in (Cascaval et al., 2017), (D'Apice et al., 2018a), (D'Apice et al., 2010a), (D'Apice et al., 2014), (D'Apice et al., 2016), (D'Apice et al., 2018b), (D'Apice et al., 2008), (D'Apice et al., 2010b), (D'Apice et al., 2012) and (D'Apice et al., 2013).

Simulations are useful to prove the proposed approach. In particular, two different choices of distribution coefficients are considered: Results obtained by the optimization approach; Random coefficients, namely: at the beginning of the simulation process, values of traffic coefficients are randomly chosen and then kept constant during the simulation. For the case study of a part of the Caltanissetta urban network in Italy, the choice of optimal distribution parameters at 2×2 nodes allows to get more suitable performances on the network. Finally, following an algorithm described in (Bretti and Piccoli, 2008) to trace car trajectories on networks, other simulations are run to test if distribution coefficients provide variations of the total travelling time for police vehicles. It is shown that times to cover a path of a single police car decrease when optimal coefficients are used.

The paper is structured as follows. Section 2 describes the model for car traffic. Section 3 considers the cost functional for police cars and the optimization of traffic coefficients. Section 4 focuses on the simulations for the case study. Conclusions end the paper in Section 5.

2. A model for car traffic

A road network is a couple $(\mathcal{N}, \mathcal{R})$, where $\mathcal{N}$ and $\mathcal{R}$ indicate, respectively, the set of nodes and roads, seen as intervals $[\eta_i, \theta_i] \subset \mathbb{R}$, $i = 1, \dots, M$. For each road, the evolution of traffic is described by the conservation law ((Lighthill and Whitham, 1955; Richards, 1956), Lighthill-Whitham-Richards model):

$$(D)_t + (f(D))_t = 0, \qquad (1)$$

where $D = D(t,x) \in [0, D_{\max}]$ represents the density of cars with $D_{\max}$ the highest possible density; $f(D) = Dv(D)$ indicates the flux with $v(D)$ the average velocity. Assuming $v_{\max} = D_{\max} = 1$, a possible decreasing velocity function is:

$$v(D) = 1 - D, \quad D \in [0,1], \qquad (2)$$

from which we get:

$$f(D) = D(1-D), \quad D \in [0,1]. \qquad (3)$$

Traffic modelling at nodes is solved via Riemann Problems (RPs), i.e. Cauchy Problems with a constant initial datum for incoming and outgoing roads.

Fix a node J of $n \times m$ type (n incoming roads I_u, $u = 1, \dots, n$; m outgoing roads, I_v, $v = n+1, \dots, n+m$) and indicate by $D_0 = (D_{1,0}, \dots, D_{n,0}, D_{n+1,0}, \dots, D_{n+m,0})$ the initial datum.

A Riemann Solver (RS) for J is a map $RS : [0,1]^n \times [0,1]^m \to [0,1]^n \times [0,1]^m$ that associates to D_0 a vector $\widehat{D} = \left(\widehat{D}_{1,0}, \dots, \widehat{D}_{n,0}, \widehat{D}_{n+1,0}, \dots, \widehat{D}_{n+m,0}\right)$ so that the the wave $\widetilde{D}_u = \left(D_{u,0}, \widehat{D}_u\right)$ is solution for an incoming road I_u, $u = 1, \dots, n$, while the wave $\widetilde{D}_v = \left(\widehat{D}_v, D_{v,0}\right)$ is solution for an outgoing road I_v, $v = n+1, \dots, n+m$. For RS, the following conditions must hold: (C1) $RS(RS(D_0)) = RS(D_0)$; (C2) for I_u, $u = 1, \dots, n$ (resp. I_v, $v = n+1, \dots, n+m$), the wave $\widetilde{D}_u$ (resp. $\widetilde{D}_v$) has negative (resp. positive) speed.

If $n \leq m$, a possible RS at node J is defined by ((Coclite et al., 2005)):

- **(A)** Traffic distributes at J by some coefficients, collected in a traffic distribution matrix $A = (\alpha_{v,u})$, $u = 1, \dots, n$, $v = n+1, \dots, n+m$, $0 < \alpha_{v,u} < 1$, $\sum_{v=n+1}^{n+m} \alpha_{v,u} = 1$. The u-th column of A gives the percentage of traffic that, from the incoming road I_u, goes to the outgoing roads.
- **(B)** Respecting (A), drivers maximize the flux through J.

If $m < n$, a further rule (yielding criterion) is necessary:

- **(C)** If W is the maximal amount of cars that can enter the outgoig roads, then $p_u W$ cars come from I_u, where $p_u \in]0,1[$, $\sum_{u=1}^{n} p_u = 1$, is the right of way parameter for I_u, $u = 1, \dots, n$.

2.1. Two incoming roads and two outgoing roads

For a node J of 2×2 type (incoming roads I_1 and I_2; outgoing roads I_3 and I_4), indicate the densities of cars for incoming and outgoing roads, respectively, by $D_u(t,x) \in [0,1]$, $(t,x) \in \mathbb{R}^+ \times I_u$, $u = 1,2$, and $D_v(t,x) \in [0,1]$, $(t,x) \in \mathbb{R}^+ \times I_v$, $v = 3,4$. From (C2), for the flux (3) and initial datum $D_0 = (D_{1,0}, D_{2,0}, D_{3,0}, D_{4,0})$ for the node J, we prove that the maximal flux values on roads are:

$$\gamma_\phi^{\max} = \begin{cases} f(D_{\phi,0}), & \text{if } 0 \le D_{\phi,0} \le \frac{1}{2} \text{ and } \phi = 1,2, \\ & \text{or } \frac{1}{2} \le D_{\phi,0} \le 1 \text{ and } \phi = 3,4, \\ f\left(\frac{1}{2}\right), & \text{if } \frac{1}{2} \le D_{\phi,0} \le 1 \text{ and } \phi = 1,2, \\ & \text{or } 0 \le D_{\phi,0} \le \frac{1}{2} \text{ and } \phi = 3,4. \end{cases}$$

In this case, matrix A has the coefficients $\alpha_{3,1}$, $\alpha_{3,2}$, $\alpha_{4,1} = 1 - \alpha_{3,1}$, $\alpha_{4,2} = 1 - \alpha_{3,2}$, and the assumption $\alpha_{3,1} \neq \alpha_{3,2}$ guarantees the uniqueness of solutions.

From rules (A) and (B), the flux solution to the RP at J, $\widehat{\gamma} = (\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\gamma}_4)$, is found as follows: the outgoing fluxes are $\widehat{\gamma}_v = \alpha_{v,1}\widehat{\gamma}_1 + \alpha_{v,2}\widehat{\gamma}_2$, $v = 3,4$; the incoming fluxes $\widehat{\gamma}_u$, $u = 1,2$, are solutions of the linear programming problem max $(\gamma_1 + \gamma_2)$, with $0 \le \gamma_u \le \gamma_u^{\max}, u = 1,2$, $0 \le \alpha_{3,1}\gamma_1 + \alpha_{3,2}\gamma_2 \le \gamma_3^{\max}$, $0 \le (1-\alpha_{3,1})\gamma_1 + (1-\alpha_{3,2})\gamma_2 \le \gamma_4^{\max}$.

Once $\widehat{\gamma}$ is known, $\widehat{D}$ is found as:

$$\widehat{D}_\phi \in \begin{cases} \{D_{\phi,0}\} \cup]\varepsilon(D_{\phi,0}), 1], & \text{if } 0 \le D_{\phi,0} \le \frac{1}{2} \text{ and } \phi = 1,2, \\ & \text{or } \frac{1}{2} \le D_{\phi,0} \le 1 \text{ and } \phi = 3,4, \\ \left[0, \frac{1}{2}\right], & \text{if } 0 \le D_{\phi,0} \le \frac{1}{2},\ \phi = 3,4, \\ \left[\frac{1}{2}, 1\right], & \text{if } \frac{1}{2} \le D_{\phi,0} \le 1,\ \phi = 1,2, \end{cases}$$

where $\varepsilon : [0,1] \to [0,1]$ is the map such that $f(\varepsilon(D)) = f(D)\ \forall\, D \in [0,1]$ and $\varepsilon(D) \neq D\ \forall\, D \in [0,1] \setminus \left\{\frac{1}{2}\right\}$.

2.2. Numerical approximations

To define possible numerical approximations for the density $D(t,x)$ on the roads of a traffic network, we deal with the Godunov scheme (see (Godunov, 1959)). Assume the flux (3) and consider a numerical grid with: Δx, space grid size on each road; Δt, time grid size on the time interval $\left[0, \overline{T}\right]$; K and H, respectively, the number of time and space nodes of the grid; $(t_k, x_h) = (k\Delta t, h\Delta x)$, for $k = 0,1,\ldots,K$ and $h = 0,1,\ldots,H$, the grid points.

Using the notation $D_h^k = D(t_k, x_h)$, the Godunov scheme is expressed as, for $k = 0,1,\ldots,K-1$, $h = 0,1,\ldots,H$:

$$D_h^{k+1} = D_h^k - \frac{\Delta t}{\Delta x}\left(g^G\left(D_h^k, D_{h+1}^k\right) - g^G\left(D_{h-1}^k, D_h^k\right)\right),$$

where Δx and Δt satisfy the CFL condition $\Delta t \le \frac{\Delta x}{2}$, while $g^G(a,b)$ is the numerical flux given by:

$$g^G(a,b) = \begin{cases} \min(f(a), f(b)), & a \le b, \\ f(a), & b < a < \frac{1}{2}, \\ f\left(\frac{1}{2}\right), & b < \frac{1}{2} < a, \\ f(b), & \frac{1}{2} < b < a. \end{cases}$$

For incoming roads not linked on the left and outgoing roads not linked on the right, boundary conditions are necessary. Hence, for roads connected at the right endpoint, the interaction at a node is given by:

$$D_H^{k+1} = D_H^k - \frac{\Delta t}{\Delta x}\left(\widehat{\gamma}_u - g^G\left(D_{H-1}^k, D_H^k\right)\right);$$

for roads connected at the left endpoint, we consider:

$$D_0^{k+1} = D_0^k - \frac{\Delta t}{\Delta x}\left(g^G\left(D_0^k, D_1^k\right) - \widehat{\gamma}_v\right),$$

where $\widehat{\gamma}_u$ and $\widehat{\gamma}_v$ are the solutions of RSs at nodes.

3. Optimal coefficients for traffic dynamics

Assume that some gathering phenomena occur on some parts of an urban network and that police cars need to reach the places where crowds are. For the police cars, the following velocity function is assumed:

$$\varphi(D) = 1 - \chi + \chi v(D), \tag{4}$$

where $0 < \chi < 1$ and $v(D)$ obeys (2). As $\varphi(D_{\max}) = 1 - \chi > 0$, then the police vehicles have higher velocities than cars. For a node J with incoming roads I_1 and I_2 and outgoing roads I_3 and I_4, for a fixed initial datum $(D_{1,0}, D_{2,0}, D_{3,0}, D_{4,0})$, the cost functional $V_{(u,v)}(t)$, that represents the average velocity of police cars that cross the incoming road I_u, $u \in \{1,2\}$, and the outgoing road I_v, $v \in \{3,4\}$, is defined as:

$$V_{(u,v)}(t) := \int_{I_u} \varphi(D_u(t,x))\,dx + \int_{I_v} \varphi(D_v(t,x))\,dx. \tag{5}$$

If $u = 1$ and $v = 3$, we get the following theorem (the statement is straightforward for different cases of u and v).

Theorem 1 *Consider a node J with incoming roads I_1 and I_2, and outgoing roads I_3 and I_4. For a time $t >> 0$, the coefficients $\alpha_{3,1}$ and $\alpha_{3,2}$, that maximize $V_{(1,3)}(t)$, are $\alpha_{3,1}^{OPT} = \frac{\gamma_1^{\max} - \gamma_4^{\max}}{\gamma_1^{\max}}$, $0 \le \alpha_{3,2}^{OPT} < \alpha_{3,1}^{OPT}$, with the following exceptions, for which the optimal values do not exist and are approximated as: for ϵ_1 and ϵ_2 small, positive and such that $\epsilon_1 \neq \epsilon_2$, $\alpha_{3,1}^{OPT} = \epsilon_1$, $\alpha_{3,2}^{OPT} = \epsilon_2$ if $\frac{\gamma_1^{\max}}{\gamma_4^{\max}} \le 1$; $\alpha_{3,1}^{OPT} = \frac{\gamma_3^{\max}}{\gamma_3^{\max} + \gamma_4^{\max}} - \epsilon_1$, $\alpha_{3,2}^{OPT} = \frac{\gamma_3^{\max}}{\gamma_3^{\max} + \gamma_4^{\max}} - \epsilon_2$ if $\frac{\gamma_3^{\max} + \gamma_4^{\max}}{\gamma_1^{\max}} < 1$.*

Remark 2 *Proof of Theorem 1 is straightforward considering the analysis in (Cascone et al., 2007) and (Cascone et al., 2008).*

4. Simulations

The goodness of the optimization results, defined by Theorem 1, is studied on a real network by different simulation choices. This analysis is then completed by computing the travelling times of a police car on assigned paths.

We consider a part of the network of Caltanissetta, Italy (see Figure 1).

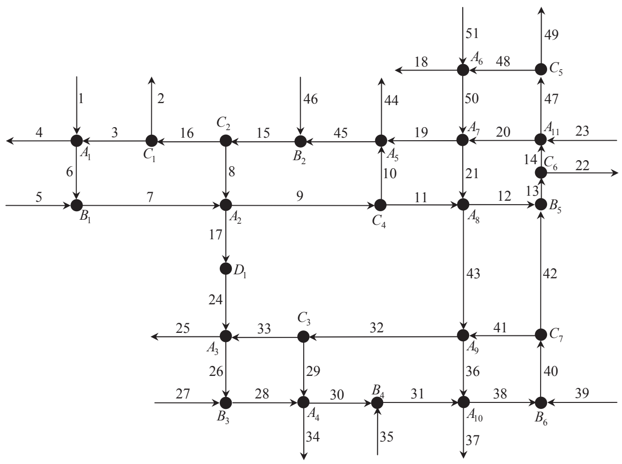

Figure 1. Topology of a part of the network of Caltanissetta, Italy.

Table 1. Numbers and roads of Figure 1.

Road	Graph segments
Via Giuseppe Mulè	1, 2
Via Luigi Monaco	3 – 21
Via della Regione	22, 23
Via Due Fontane	24 – 33
Via SD1	34, 35
Via Leone XIII	36 – 43
Via Luigi Russo	44, 45, 46
Via Poggio S. Elia	47 – 51

The network has: 8 roads, defined by different 51 segments (Table 1). Eight segments (1, 5, 23, 27, 35, 39, 46, 51) refer to incoming roads; nine segments (2, 4, 8, 22, 25, 34, 37, 44, 49) identify outgoing roads . Finally, the network presents 25 nodes of various types: 2×2, indicated by A_i, $i = 1, \dots, 11$; 2×1, represented by B_i, $i = 1, \dots, 6$; 1×2, labelled by C_i, $i = 1, \dots, 7$; 1×1, D_1. We assume that police cars could cross the path $\rho = \Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4$, with:

$$\Phi_1 = \{23, 47, 48, 50, 19, 45, 15\}, \quad \Phi_2 = \{16, 3, 6, 7, 17, 24, 26\},$$
$$\Phi_3 = \{28, 30, 31, 38, 40, 42, 13\}, \quad \Phi_4 = \{14, 20, 21, 43, 32, 33, 35\}.$$

We consider the behaviour of $S(t) = \sum_{(u,v)\in\Omega} V_{(u,v)}(t)$, with $V_{(u,v)}(t)$ defined by (5) and:

$$\Omega := \left\{ \begin{array}{l} (23,47), (48,50), (50,19), (19,45), \\ (3,6), (7,17), (24,26), (28,30), \\ (31,38), (20,21), (43,32), (33,25) \end{array} \right\}.$$

Traffic flows simulations are made by the Godunov method with $\Delta x = 0.025 = 2\Delta t$ in a time interval $[0, T]$, where $T = 100$ min. For densities, boundary data and initial conditions are chosen approaching $D_{\max} = 1$ in order to simulate a congestion scenario on the network, as follows: initial datum equal to 0.88 for all roads; boundary data 0.93 for roads 1, 5, 23, 27 and 35; 0.92 for roads 39, 46, and 51; 0.88 for roads 2, 4, 18, 22 and 25; 0.93 for roads 34, 37, 44 and 49. Considering some measures on the real network, we have, for nodes B_i, $i = 1, \dots, 6$, the right of way parameters: $p_{12} = p_{26} = 0.25$, $p_{46} = 0.35$, $p_6 = p_{35} = 0.45$, $p_{38} = p_{39} = 0.45$, $p_5 = p_{30} = 0.65$, $p_{45} = 0.75$, $p_{42} = p_{27} = 0.85$; for nodes C_i, $i = 1, \dots, 7$, the distribution coefficients: $\alpha_{41,40} = 0.25$, $\alpha_{49,47} = \alpha_{22,13} = 0.35$, $\alpha_{8,15} = \alpha_{33,32} = 0.45$, $\alpha_{2,16} = \alpha_{3,16} = \alpha_{10,9} = \alpha_{11,9} = 0.5$, $\alpha_{16,15} = \alpha_{29,32} = 0.65$, $\alpha_{48.47} = \alpha_{14,13} = 0.7$, $\alpha_{42,40} = 0.85$. Finally, $\chi = 0.5$ is used.

For simulations, we analyze two different cases: locally optimal distribution coefficients (*optimal case*) for each node A_i $i = 1, \dots, 11$, i. e. parameters that deal with Theorem 1; random coefficients (*random case*), namely the distribution parameters are chosen randomly at each node A_i $i = 1, \dots, 11$ when the simulation starts and then are kept constant.

Figure 2 depicts the behaviour of $S(t)$. The optimal simulation is indicated by a continuous curve, and random cases by dashed lines. As expected, random simulations of $S(t)$ are lower than the optimal behaviour. Precisely, when optimal parameters are considered, nodes of 2×2 type have congestion reductions due to the redistribution of flows on roads. Even if right of way parameters of nodes B_i, $i = 1, \dots, 6$, and distribution coefficients of nodes C_i, $i = 1, \dots, 7$, are used by the results of (Cascone et al., 2007) and (Cascone et al., 2008), traffic conditions are almost unaffected.

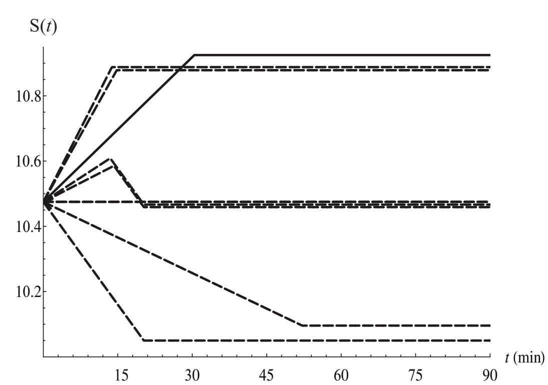

Figure 2. $S(t)$ in $[0, 90]$ using optimal coefficients (continuous line) and random parameters (dashed lines).

Suppose that a police car crosses a path in a network. Its position $z = z(t)$ is modelled by the Cauchy problem:

$$\begin{cases} \dot{z} = \varphi(D(t,x)), \\ z(t_0) = z_0, \end{cases} \quad (6)$$

where z_0 is the initial position at the initial time t_0. A numerical method (see (Bretti and Piccoli, 2008)) allows a possible estimation of the travelling time of the police car. We compute the trajectory along road 24 and the time to cover it in optimal and random conditions.

In Figure 3, we assume that the police car starts its travel at the beginning of road 24 at the initial time $t_0 = 70$ and compute the trajectories $z(t)$ in optimal (continuous line) and random cases (dashed lines).

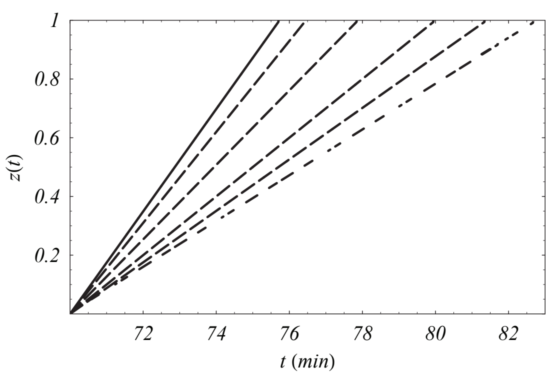

Figure 3. Trajectory $z(t)$ for a police car along road 24 with $t_0 = 70$; optimal parameters (continuous line) and random coefficients (dashed lines).

The behaviour $z(t)$ in the optimal case has always a higher slope than the trajectories in random cases as traffic levels are low. When random parameters are used, shocks propagating backwards increase the density values on the network; The velocity for the police car is reduced and exit times from road 24 become longer. Assuming $t_0 = 70$ we have the following time instants t_{out} in which the police car goes out of road 24, either for the optimal distribution coefficients (t_0^{opt}) or random choices ($t_0^{r_i}$, r_i, $i = 1,..,5$): $t_0^{opt} = 75.56$, $t_0^{r_1} = 76.52$, $t_0^{r_2} = 77.88$, $t_0^{r_3} = 79.79$, $t_0^{r_4} = 81.47$ and $t_0^{r_5} = 82.53$.

5. Conclusions

This paper deals with an optimization study whose aim is to face gathering phenomena. Optimal distribution coefficients at road junctions with two incoming and two outgoing roads are achieved by maximizing a cost functional, that represents the average velocity of police vehicles. Simulations on a real urban network prove the goodness of the optimization procedure as well as, in case of gathering conditions, the possibility of fast transits through the estimation of the trajectories of emergency vehicles. Future research activities foresee the extension of the proposed issue by either different cost functionals or other optimization approaches, mainly based on genetic algorithms.

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